A time–dependent model of the Earth's magnetic field and its secular variation for the period 1980 to 2000

I. Wardinski

Section 2.3, Earth's Magnetic Field, GeoForschungsZentrum Potsdam,

Germany

R. Holme

Department of Earth and Ocean Sciences, University of Liverpool, Liverpool,

United Kingdom

I. Wardinski, Section 2.3, Earth's Magnetic Field, GeoForschungsZentrum Potsdam, Telegrafenberg Building F428, Potsdam, D-14473, Germany. (ingo@gfz-potsdam.de)

Abstract. This study presents an investigation and description of the secular variation of the Earth's magnetic field between 1980 and 2000. A time–dependent model, C^{3}FM (Continuous Covariant Constrained-end-points Field Model), of the main field and its secular variation between 1980 and 2000 is developed, with Gauss coefficients expanded in time on a basis of cubic B–splines. This model is constrained to fit field models from high quality vector measurements of MAGSAT in 1980 and Ørsted in 2000 and to fit both magnetic observatory and repeat station secular variation estimates for the period in between. These secular variation estimates (first time derivatives) are derived from observatories monthly or annual means and repeat station data in order to reduce the contributions of crustal noise, annual and semi–annual variation. On average, the model input consists secular variation estimates of the X, Y and Z components at 130 locations per month. Treatment of covariance between the different components allows a higher temporal sensitivity of the model, due to the exclusion of some external field variation. The model is computed up to degree and order 15.

The model is a useful extension of the hitherto existing time–dependent description of the secular variation, GUFM which describes the secular variation until 1990. It reveals a short term secular variation on sub-decadal time scale and has a higher spatial resolution, than previously resolved. The model is also valuable to test the frozen flux hypothesis and to link features of the radial field at the core–mantle boundary to the geodynamo.

1. Introduction

This paper is a development of earlier work in mapping the magnetic field at the core–mantle boundary (CMB) by *Bloxham and Jackson* [1992] and *Jackson et al.* [2000]. Previously, *Jackson et al.* [2000] provided a continuous description of the Earth's main magnetic field and its temporal change for the period 1590 to 1990. Here, we apply a similar concept to derive a continuous model for the time interval from 1980 and 2000. This time span is bracketed by the availability of high quality measurements of the Earth's magnetic field from satellite missions. In 1980 the MAGSAT mission provided for the first time vector measurements of the geomagnetic field with high spatial resolution, over a period of six months. The next mission of comparable quality was the Ørsted mission, launched in 1999 [*Neubert et al.*, 2001], followed by the CHAMP mission, launched in 2000 [*Reigber et al.*, 2002]. Both satellites have provided high–quality vector measurements of the geomagnetic field. Satellites provide global data coverage, but with limited control of the temporal coverage. Different data points are recorded at different local times, and therefore have different contributions from magnetic variations such as the daily variation. At ground observatories, external field variation can to some extent be argued to average out over time, but with satellite data, this is not possible, as it is rare to return to exactly the same location. The best modelling can be attempted by fitting a global model for quiet-time data, also with various aspects of the external field parameterized, perhaps in terms of Dst or an equivalent index. Secular variation (the time variation of the field) can be solved for; however, because of the non-uniform nature of the external field, and the short time period over which the satellites fly, the secular variation is much less well-determined than the field itself. In contrast, observatory data provide excellent temporal coverage, and because recordings are made at a single location, averaging (over minutes, days or years) can reduce greatly any zero-mean noise. However, their coverage is far from uniform over the globe, and the field contains a strong component of the short-wavelength crustal field (and also potentially very short wavelength induced field from induction of time-varying external field).

Two approaches are of particular note in trying to accommodate both data types. The first is the comprehensive approach of *Sabaka et al.* [2002, 2004], where the field, internal and external, is parameterized in terms of many parameters (in the tens of thousands) and the various contributions co-estimated at all length and time scales. In contrast, *Bloxham and Jackson* [1992], and following them, *Jackson et al.* [2000] included both satellite and observatory data in an simple inversion for main field only. However, as described by *Bloxham and Jackson* [1992], they had to limit the number of satellite data in the inversion, so as to limit correlation in the data misfits from lithospheric field.

We choose instead a hybrid approach, whereby we attempt to use both the satellite and observatory data to provide the most information about the field, while avoiding the computational overhead and complications of comprehensive modelling. We adopt specialized field models for the satellite epochs. These have been constructed from satellite data alone, include a parameterization of the external field appropriate to satellite orbits, and solve directly for the shorter wavelength (higher degree) lithospheric field, avoiding possible correlated errors from this source. We thus make use of all the data processing and expertise that has gone into producing the best possible field model at a satellite epoch. We then include these field models as end-constraints on the interval 1980-2000.

Between these epochs, observatory data as annual differences of monthly means are included, thereby eliminating (by averaging) many sources of contamination from external field.

The data set and the preprocessing of the data are described in section 2 and the new methodology is outlined in section 3 and 4. Section 5 discuss the results and their implications for the geodynamo.

2. Data

The data used in this work are magnetic observatory monthly means, annual means, and repeat station measurements. Where available, we use quiet-time monthly means, defined as the average at each observatory of the five international quiet days in each month. These data were obtained from a monthly mean database collated by Dr. Mioara Mandea (pers. comm.). When such quiet-time means were unavailable, then full monthly means, defined as being the average over all days of the month and all times of the day, were derived from hourly mean values downloaded from the World–Data–Center Copenhagen[1] and from the Intermagnet database[2]. Where such data were also unavailable, we used annual means provided by the British Geological Survey[3].

The repeat survey data considered in this study are taken from a database maintained by the British Geological Survey (BGS)[4]. In this study, the interval between two occupations is not longer than seven years; any occupation period beyond that is concluded not to be useful to uncover short term secular variation. Although the coverage and regular occupation of repeat stations across the European mainland are much better than in any other part of the world, these data are not considered in this study. We would not expect any improvements in recovering the secular variation on core–field length scales as there already is a good coverage of observatories in this region.

For the field modelling, three components, northward (X), eastward (Y) and downward (Z), were compiled for each observatory and repeat station. Where previously reported, the data are corrected for baseline jumps.

2.1. Secular variation estimates

In order to reduce the bias from local crustal magnetic fields, and the amplitude of the semi–annual and annual signal from external field sources and induction, rather than modelling absolute field values, we follow *Bloxham and Jackson* [1992] in estimating the value of the secular variation at each site. These estimates are calculated by taking annual differences. For example, for monthly means of the northward component

$$dX/dt|_t = X(t+6) - X(t-6) \tag{1}$$

where t denotes a particular month. Similarly, observatory annual means are treated using

$$dX/dt|_{t+1/2} = X(t) - X(t-1) \tag{2}$$

where t is here in years. This approach is known as the *n–step difference filter* [*Box and Jenkins*, 1976; *Priestley*, 1981]. It eliminates the crustal bias, because the crustal signal should be the same for both dates $t, t-n$ and therefore cancel. Furthermore, the amplitude of even irregular annual and semi-annual variations are much reduced.

3. Method

In this section we outline our method to derive a time–dependent model of the magnetic field and its secular variation at the core-mantle boundary between 1980 and 2000. The formalism is based on the approach of *Bloxham and Jackson* [1992], but additionally constraining the model at its endpoints to fit satellite field models. We do not model external fields, and assume an insulating mantle. Then, as it is in an electrical insulator, the Earth's magnetic field **B** can be represented as the gradient of scalar potential V: $\mathbf{B} = -\nabla V$. The potential has to satisfy Laplace's equation $\nabla^2 V = 0$, so the solution for internal sources in spherical geometry can be written:

$$V(t) = a \sum_{l=1}^{\infty} \sum_{m=0}^{l} (g_l^m(t) \cos(m\phi) + h_l^m(t) \sin(m\phi)) \left(\frac{a}{r}\right)^{l+1} P_l^m(\cos\theta) \,. \tag{3}$$

a is the Earth's radius (6371.2 km) and (r, θ, ϕ) the geocentric coordinates, (θ colatitude) and $P_l^m(\cos\theta)$ are the Schmidt quasi–normalized associated Legendre functions

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (P_l^m(\cos\theta) \cos(m\phi))^2 \sin\theta \, d\theta \, d\phi = \frac{4\pi}{2l+1} \,, \tag{4}$$

with degree l and order m. The coefficients $\{g_l^m, h_l^m\}$ are the Gauss coefficients, which are expanded in time on a basis of cubic B–splines $M_n(t)$ [*de Boor*, 1978]

$$g_l^m(t) = \sum_{n=1}^{N} g_l^{mn} \, M_n(t) \,, \quad h_l^m(t) = \sum_{n=1}^{N} h_l^{mn} \, M_n(t), \tag{5}$$

where $M_n(t) > 0$ if t is within the interval t_n, t_{n+4} and is zero otherwise. We choose 13 B-splines with evenly spaced knots, including knots before and after our modelled interval [similar to *Bloxham and Jackson*, 1992]. For numerical convenience, we truncate Eq. (3) at harmonic degree and order 15.

Now following closely *Gubbins and Bloxham* [1985] and *Bloxham and Jackson* [1992], we impose some a priori information of the likeliness of the solution. One source of prior information about $\mathsf{B_r}$ at the CMB is the lower bound on the required ohmic dissipation in the core to maintain the observed surface field, derived from Maxwell's equations and

Ohm's law. *Gubbins* [1975] analysis yields

$$F(\mathbf{B}_r) = \sum_{l=1}^{\infty} \frac{(l+1)(2l+1)(2l+3)}{l} \sum_{m=0}^{l} [(g_l^m)^2 + (h_l^m)^2] \text{ for } r = c\,, \tag{6}$$

where $c = 3485.0$ km, the radius of the outer core. The spatial regularization condition is then

$$\frac{4\pi}{(t_2 - t_1)} \int_{t_1}^{t_2} F(\mathbf{B}_r)\, dt = \mathsf{m}^{\mathsf{T}} N_S^{-1} \mathsf{m}\,, \tag{7}$$

where m is a vector of the Gauss coefficients. To constrain the temporal behaviour of the solution, the norm

$$\frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \int_{CMB} (\partial^2 \mathsf{B_r}/\partial t^2)^2 \, dS\, dt = \mathsf{m}^{\mathsf{T}} N_T^{-1} \mathsf{m} \tag{8}$$

is minimized, with

$$\int_{CMB} (\partial^2 \mathsf{B_r}/\partial t^2)^2 dS = \sum_{l=1}^{\infty} (l+1) \left(\frac{a}{c}\right)^{2l+4} \sum_{m=0}^{l} [(\ddot{g}_l^m)^2 + (\ddot{h}_l^m)^2] \tag{9}$$

Finally, the model covariance matrix is given by

$$\mathbf{C_m}^{-1} = \lambda_s N_S^{-1} + \lambda_t N_T^{-1}\,, \tag{10}$$

with damping parameters λ_s and λ_t. We initially followed *Backus* [1988] in setting

$$\lambda_s = 3 \times 10^{-18} \mathrm{nT}^2.$$

In practice, this condition is so weak that with the inclusion of the end-point constraints described below, it could be neglected. Aside from the weaker spatial regularization this is identical to the method outlined by *Bloxham and Jackson* [1992].

3.1. Extension of the Bloxham–Jackson formalism

The essential new aspect of this study is the utilization of a priori information for the geomagnetic field at the endpoints 1980 and 2000. At the endpoints the time–dependent field model has to fit a priori field models (m_0^{1980} and m_0^{2000}) derived from satellite vector data [*Cain et al.*, 1989; *Olsen*, 2002]. These models are originally derived to spherical harmonic degree 66 and 27, respectively. In order to construct models for the endpoints the original models are first truncated to degree 15, and tapered as follows.

We seek field models that are smooth at the CMB, but would fit the data at the Earth's surface. We choose to minimize

$$\Gamma = \int_{r=a} (\mathbf{B} - \mathbf{B}_S)^2 dS + \mu \int_{CMB} \mathbf{B}^2 dS \tag{11}$$

where μ is a Lagrange multiplier and $\mathbf{B}_S$ is the accurate model from satellite data including crustal field. Spherical harmonic orthogonality allows simplification of this expression to

$$\Gamma = \sum_{l,m} (l+1) \left((g_l^m - g_{l\ 0}^{m})^2 + (h_l^m - h_{l\ 0}^{m})^2 \right) + \mu (l+1) \left(\frac{a}{c} \right)^{(2l+4)} \left((g_l^m)^2 + (h_l^m)^2 \right) \tag{12}$$

By standard methods, the Gauss coefficients are therefore given by

$$g_l^m = \frac{g_{l\ 0}^{m}}{1 + \mu \left(\frac{a}{c} \right)^{(2l+4)}} \tag{13}$$

Therefore high-degree coefficients are damped much more than the low degree ones. The degree of fall-off (tapering) is controlled by μ, which is chosen to be effective at about $l = 12$ and above.

Then the a priori model regularization condition is

$$\int_{r=a} (\mathbf{B} - \mathbf{B}_0)^2 dS|_{t=1980} = \sum_{l=1}^{\infty} (l+1) \sum_{m=0}^{l} [(g_l^m - g_{l\ 0}^{m})^2 + (h_l^m - h_{l\ 0}^{m})^2]_{t=1980} \tag{14}$$

where m^{1980} is the model vector of the Gauss coefficients for the particular epoch 1980.0. A similar expression follows for the constraint at $t = 2000.0$. The combined end-point constraints are then written

$$\mathbf{C_a}^{-1} = \lambda_1 N_a^{-1} + \lambda_2 N_a^{-1} \, . \tag{15}$$

with damping parameters λ_1, λ_2 for 1980.0 and 2000.0.

The full objective function is therefore

$$\Theta(m) = (\mathsf{y} - \mathbf{A}\mathsf{m})^{\mathsf{T}} \mathbf{C_e}^{-1} (\mathsf{y} - \mathbf{A}\mathsf{m}) + \mathsf{m}^{\mathsf{T}} \mathbf{C_m}^{-1} \mathsf{m} + \mathsf{m}^{\mathsf{T}} \mathbf{C_a}^{-1} \mathsf{m} \,. \tag{16}$$

The damping controlling the departure from the a priori models is chosen so that the endpoint models for 1980.0 and 2000.0 are very close to the tapered satellite main field models. However, we do not force equality, because the tapering is a numerical procedure only, and therefore we do not wish the model to fit the end-point models exactly if this contradicts the observatory data. In practice, however, changing these parameters within a range that ensures a reasonably close fit to the end-point models does not greatly affect the solution.

4. Modelling and selection of temporal damping parameter

The data are modelled by applying iteratively re-weighted least squares. Our algorithm comprises of eight steps:

1. A initial model is computed weighting all data with the same uncertainty (5 nT/yr).

2. The deviations of the data from the initial model are calculated, and adopted as new weights for the data. Data with large scatter from the model are therefore down-weighted.

3. A model is derived from the newly (re-) weighted data set.

4. Data are discarded which deviate by more then 2 σ from the second model.

5. A interim model is derived from this reduced data set.

In steps 6 to 8 we generalized step 2 (the estimation of the data weighting) by further considering the covariance between the different secular variation residuals in the (X,Y,Z) directions at each location, in order to allow for possible correlated errors. For example, in mid-latitudes, we might expect that the error would be dominated by the unmodeled signal from external field variations, in particular the ring current, leading to a particularly strong error correlation between the X and Z components. We assume that the covariance is stationary over the 20-year period, and therefore constructed 3x3 error covariance matrices for each location. The data covariance matrix $\mathbf{C_e}$ is then block diagonal with 3x3 blocks for each site. Its inversion is straightforward by inverting each of the 3x3 matrices, after which it is applied in (16) in order to yield a final model. The application of correlation between the three vector components at a particular location was first considered in the modelling of attitude error in vector magnetic satellite data [*Holme and Bloxham*, 1996].

Although we have potentially five damping parameters $(\mu, \lambda_s, \lambda_t, \lambda_1, \lambda_2)$, as discussed above the most important one (giving the largest variation of the model for the parameter range in which we are interested) is λ_t, controlling the temporal damping. Models are derived for a range of temporal damping parameters $3.5 \cdot 10^2 \leq \lambda_t \leq 3.5 \cdot 10^{-5}$. The appropriate temporal damping was chosen by considering a trade-off curve – a log-log plot of misfit vs. the temporal norm – with the optimal solution chosen at the "knee" of the curve. The parameters of the selected model are summarized in Table 1.

A preliminary version of this model at epoch 1995.0 was submitted as a candidate model for DGRF95, and contributed to the final averaged model selected [*Macmillan et al.*, 2003].

5. Results and Discussion

Before presenting results we would like to discuss what advantages the new methodology gives over the methods of *Bloxham and Jackson* [1992] and *Jackson et al.* [2000] for an interval bounded by high-resolution field models from satellite data. In order to discuss the differences between GUFM and C^{3}FM the power spectra of the main field and secular variation at the CMB are derived for 1980 and 1990, epochs at which the models overlap. The power spectrum is defined by

$$\langle \mathbf{B}^2 \rangle = \sum_{l=1}^{\infty} \left(\frac{a}{c} \right)^{2l+4} (l+1) \sum_{m=0}^{l} (g_l^{m2} + h_l^{m2}) \tag{17}$$

[*Lowes*, 1966]. It quantifies the spectral energy of the single spherical harmonic degrees of the model. The spectra of the main field for different epochs, which are shown in Fig. 1, are similar up to degree 11. Above this, they differ, due to the different methods of controlling small-scale field structure. The GUFM spectra fall off due to spatial regularization, whereas the high degree structure in our model is controlled by the tapered end-point models in combination with the temporal damping, with the effect of tapering dominating above degree 13. In principle, there is no difference between these two methods: both involve damping of high-degree structure. However, for GUFM, the damping had to be chosen to be appropriate for non-satellite data epochs, with inherent lower resolving power than when satellite data are available. Satellite data were also culled so as to avoid correlated errors from crustal field. Our model assumes that correlation has been dealt with by direct parameterization in the satellite models. Our choice of tapering is more subjective, based on a conservative estimate of the degree at which the crustal field is a significant fraction of the total internal field. Thus, that our model has higher spatial resolution depends fundamentally on this choice, but the inclusion of all available satellite data to form the end models allows this choice to be made higher than would be possible with the less numerous data set required for GUFM.

We also consider a spectrum for secular variation, defined in analogy with the main field (17) as

$$\langle \dot{\mathbf{B}}^2 \rangle = \sum_{l=1}^{\infty} \left(\frac{a}{c}\right)^{2l+4} (l+1) \sum_{m=0}^{l} (\dot{g}_l^{m2} + \dot{h}_l^{m2}). \tag{18}$$

The spectra of the secular variation of GUFM and C^{3}FM match closely up to degree 6, see Fig. 2, although note that there is much greater variation between epochs in our model than in GUFM. This is a result of the higher temporal density of the basis functions, and also the treatment of covariance in the data providing stronger constraint on the SV, discussed below. Thereafter, the power of the C^{3}FM is greater than the power of GUFM spectra by a factor 3. Above degree 7 damping becomes important and controls the GUFM spectra from degree 9. For our C^3 model, the no-penalty condition for the time variation of the field is uniform secular variation between the end-point models: this is seen in the identical SV spectra for the two epochs at degrees 13-15.

We therefore claim that our model has additional content over GUFM arising from our methodology, with a marginal improvement in temporal resolution at low harmonic degree, and an increase of approximately two spherical harmonic degrees for the main field and about five spherical harmonic degrees for the secular variation for the point at which the model is totally controlled by damping.

Error and resolution analysis

Fig. 3 focuses on detailed analysis of the fit of the model to a particular data series, that from Niemegk observatory. Fig. 3(a) shows the residuals between the modelled and observed secular variation in all components. Here, the $\dot{\mathrm{Z}}$ residuals show the largest amplitude, and the $\dot{\mathrm{Y}}$ residuals the smallest. Note the anticorrelaton between the $\dot{\mathrm{X}}$ and $\dot{\mathrm{Z}}$ residuals, as would be expected from unmodeled ring current.

The residuals are further analysed by means of autocorrelation and cross-correlation functions, given in Fig. 3(b) and 3(c). The autocorrelation function is defined as

$$A(\tau) = \frac{1}{N-\tau} \sum_{t=1}^{N-\tau} (x_t - \langle x \rangle)(x_{t+\tau} - \langle x \rangle)/(x_t - \langle x \rangle)^2 \,, \tag{19}$$

where x_t is time series of the residuals (i.e. in $\dot{\mathrm{X}}$), $\langle x \rangle$ is their mean, and τ is the shift. A strong negative peak is seen in all three components at 12 months, due to the use of annual first differences to compute the secular variation estimates (cf. (1)). There is no further obvious structure in the autocorrelation function. We therefore conclude that the model is capable of explaining most if not all of the available secular variation signal in the data.

The cross correlation function of the residuals against each other is shown in Fig. 3(c) and is defined to be

$$C(\tau) = \frac{1}{N-\tau} \frac{\sum_{t=1}^{N-\tau} (x_t - \langle x \rangle)(y_{t+\tau} - \langle y \rangle)}{\sqrt{(\sum_t (x_t - \langle x \rangle)^2)}\sqrt{(\sum_t (y_{t+\tau} - \langle y \rangle)^2)}} \,. \tag{20}$$

It shows the mutual correlations of two independent time series x_t and y_t. The maxima at zero lag indicate that the variation of the residual components are correlated or anti–correlated, respectively. In the particular case for Niemegk this implies the anti-correlation of $\dot{\mathrm{X}}$ and $\dot{\mathrm{Z}}$ noted above and also shows correlation of both components with $\dot{\mathrm{Y}}$. The only other strongly significant features are at $\tau = 12$ months shift, again due to the data processing.

The clear cross–covariance of the residuals in the three components provides support for our consideration of error covariance between the different field elements at one location. Figure 4 (a-c) presents the fit of the model to the three vector components recorded by Niemegk magnetic observatory (NGK). The dot-dashed line is the prediction of the model prior to consideration of model covariance (step 6 in our modelling procedure), while the solid line gives the prediction of the model obtained with covariances. The behaviour of the $\dot{Y}$ component is particularly striking. Without considering the covariances, temporal details in $\dot{Y}$ variation are very closely fit, but when the covariance is considered, the behaviour is much simpler. Why this comes about is shown in Figure 4 (d-f), which plots the predictions of both models against the data, but this time projecting in the directions of the eigenvectors of the Niemegk data covariance matrix. The eigenvalues and eigenvectors in these three directions are

$$\begin{aligned} e_1 &= (0.254, 0.966, -0.047), \;\; \lambda_1 = (1.66\mathrm{nT/yr})^2 \\ e_2 &= (-0.637, 0.131, -0.760) \;\; \lambda_2 = (2.71\mathrm{nT/yr})^2 \\ e_3 &= (-0.728, 0.223, 0.648) \;\; \lambda_3 = (9.11\mathrm{nT/yr})^2 \end{aligned} \tag{21}$$

For comparison, the simple rms misfits in $(\dot{X}, \dot{Y}, \dot{Z})$ are $(6.87, 2.61, 6.25)$ nT/yr respectively. The eigenvector with the lowest noise is predominantly made up of the $\dot{Y}$ component. The other two directions are combinations of $\dot{X}$ and $\dot{Z}$ respectively approximately perpendicular to and parallel to the mean effect of the ring current (including induced field). When the raw $(\dot{X}, \dot{Y}, \dot{Z})$ data are fit, both $\dot{X}$ and $\dot{Z}$ are noisy, but resolved into the frame of the eigenvectors, the direction perpendicular to the ring current effect has much lower noise, and therefore the model attempts a much closer fit to the data in this direction, as is clearly seen in the figure. This closer fit provides a strong constraint on model behaviour, as two components must be fit closely. This allows differentiation between internal and external sources, and so the very fine scale detail in $\dot{Y}$ is no longer fit, as to do so is inconsistent with fitting the intermediate eigenvector direction. Without this constraint, we might erroneously fit far too much detail in $\dot{Y}$ (particularly as the data are dominated in quantity and often in cleanliness by European observatories), leading to an inappropriate interpretation as to what part of the secular variation can be explained by an internal field. Hence, consideration of the error covariance allows us to fit the data more closely with reduced risk of external field contamination, thereby allowing us to take advantage of the higher resolution of our temporal basis.

Note that this problem is new: the previous UFM and GUFM models provide a rougher fit to the data, and on longer time scales (decadal variation) achieve a good separation of internal and external fields. It is only because we are attempting to fit higher resolution (shorter time scale) features that treatment of error covariance has such an important effect.

In order to estimate to what extent the model parameters are determined by the data, we compute the resolution matrix. Low resolution and inability to satisfy the data entirely are both sources of uncertainty in the model estimates. The resolution matrix is given by

$$\mathbf{R} = (\mathbf{A}^\mathsf{T}\mathbf{C_e}^{-1}\mathbf{A} + \mathbf{C_m}^{-1} + \mathbf{C_a}^{-1})^{-1}\mathbf{A}^\mathsf{T}\mathbf{C_e}^{-1}\mathbf{A}\,. \tag{22}$$

Were all model parameters perfectly resolved by the data, this matrix would be the identity matrix. Due to the non–uniqueness of the problem and data inadequacy, to obtain a solution a regularization scheme must be applied, resulting in off–diagonal elements of the model covariance matrix and limiting the resolution.

A resolution near 1 signifies that a model parameter is wholly determined by the data, whereas a low resolution means that the model parameter is mostly constrained by the a priori information. Figure 5 shows the estimates of the resolution of all 255 model parameters (the diagonal elements of the resolution matrix) for six different model epochs. The comparatively low resolution in 1980 and 2000 results from the definition of the resolution matrix. In (22), the matrix is defined as the resolution related to the secular variation estimates. "Resolution" from the end-point models is not considered, as we parameterize this separately: thus, if the model at an end epoch was determined entirely by the constraint, we would calculate a resolution of 0. At the model end points only two of the four contributing B–splines are constrained by the observatory data. Therefore, the values are less significant than the a priori models, and so the calculated resolution is lower. The step–like appearance of these curves indicates that the resolution of the coefficients within each degree is nearly equal. Only the coefficients g_8^m, h_8^m (coefficient numbers 64–80) show a slight tendency to a higher resolution with increased order. Gauss coefficients of degree and order greater than 12 (coefficient number > 168) are unresolved. This agrees with the analysis of power spectra (see Fig. 1).

6. Model - data comparison

In figure 6, we compare the observed and modelled secular variation at a number of magnetic observatories, chosen for a broad geographic distribution from polar through mid to low latitudes (see Table 2 for details). The model fits the observed secular variation well, and as a bonus enables detection of previously unreported baseline jumps: see, for example, the graphs of $\dot{\mathrm{Y}}$ and $\dot{\mathrm{Z}}$ in Hermanus around 1997. The model reveals a sub–decadal variability in all components at the mid and low latitude observatories. However, the secular variation of observatories at similar latitude, but different longitude, show different behaviour, for example Niemegk and Newport or MBour and Pamatai. Only the behaviour of Hermanus and Eyrwell show strong similarities. At all observatories periods of increasing and decreasing secular variation are delineated by sudden events where the slope of the secular variation changes its sign – the well–known geomagnetic jerks [*Alexandrescu et al.*, 1995]. These jerks are most clearly identified in the secular variation of the Y component, because this component is less influenced by external field variation. Because our model filters out noise from the external field, it is perhaps easier to identify the jerks in the model than in the data themselves. The first of the geomagnetic jerks which occurred during this period is visible around 1983 in $\dot{\mathrm{Y}}$ component of Hermanus [*Dowson et al.*, 1987; *Kotzé*, 2003]. It was originally thought that this event was only discernible in the recordings of observatories in the southern hemisphere, but the extent of this event appears much wider, reaching even a low latitude observatory, MBour. This merits further study. A jerk has been reported for 1991 [*Macmillan*, 1996], and is mainly visible in East components of European observatories such as Niemegk. However, the third jerk which occurred during this period, around 1999 [*Mandea et al.*, 2000], is not clearly visible in our model, probably because of its closeness to the interval end.

Beside the known jerks, there are further features present in the secular variation, for example around 1986 in MBour, Pamatai and Eyrwell, 1994 in MBour, and around 1997 in Niemegk and MBour. The latter event is discussed by *Sabaka et al.* [2004], who find evidence for the global extent of this feature; therefore, they classified it as a jerk.

$\dot{\mathrm{X}}$ and $\dot{\mathrm{Z}}$ show also distinct variation over this 20 years time span. It should be borne in mind that induction effects might be present in $\dot{\mathrm{Z}}$, but nevertheless, the short term variations (shorter then 5 years) caused by external field variation seem to be excluded from the model. Whereas the secular variation in X and Z of Resolute Bay and Scott Base is more or less featureless, for the other observatories it is rich in detail. All non–polar observatories show a distinct feature in $\dot{\mathrm{Z}}$ around 1982 which is also seen in $\dot{\mathrm{X}}$ of MBour and Pamatai. This feature may be associated with the 1983 jerk. Also for the

1991 jerk, there are sharp changes visible in the secular variation of X and Z for most of the observatories, leading or trailing the 1991 event by about one year.

Morphology of the radial field at the core-mantle boundary

In figure 7 the radial component of the magnetic field is shown for 4 epochs: 1980.0, 1985.0, 1995.0 and 2000.0. (1990.0 is included in figure 9.) The maps show a basically dipolar structure, overlain by localized concentrations of high flux separated by larger areas of weaker flux. There is a clear coherence between the maps at different epochs, with the observable features (e.g., patches with the opposite flux to their vicinity and bounded by a null–flux curve – reverse flux patches) equally visible at all epochs. In order to characterize these features regions of high field are marked as N (normal) and R (reversed), and numbered as given in figure 8.

The most prominent such feature is the patch R2 beneath the southern Atlantic. This reversed flux patch was identified by *Gubbins and Bloxham* [1985] and has grown continuously since it appeared around 1970 [*Jackson et al.*, 2000]. In 1984 the patch merged with a minor patch R1 underneath Antarctica in our model, and finally united with the major positive flux pattern of the northern hemisphere in 1997. This event leads to the magnetic equator almost reaching the south pole! The size, position and flux of the other reversed flux patches of the southern hemisphere remained almost unchanged during this period. There are also reversed flux patches in the north polar region. Initially a single patch R8 existed, which split into two individual patches R8A and R8B in 1986.

Fig. 9 shows a comparison of the radial field at the CMB from C^{3}FM and GUFM in order to enable a discussion of the differences between both models. On the large scale the maps of both models are very similar. The maps differ in detail, such as shape and amplitude of some high flux patches. Perhaps unexpectedly, some of the small-scale detail is stronger in GUFM than in C^{3}FM, probably associated with the peaks in the spectra at degree 9 (Figure 1). This may be related to temporal edge effects in GUFM. Although weaker in the new model, these features show more detailed structure. For example, the North polar reverse flux patch is divided in the C^{3}FM–map, whereas in GUFM this patch appears undivided. Further, a reverse flux patch west of Mexico appears in C^{3}FM and is absent in GUFM. We would expect such small scale differences between both due to different choices of damping parameter.

Testing the frozen flux hypothesis

The hypothesis that the secular variation on short time scales is entirely given by the advection term of the induction equation is known as the frozen flux hypothesis [*Alfvén*, 1942; *Roberts and Scott*, 1965]. The radial field at the core–mantle boundary has to satisfy certain conditions, such as

$$F = \int_S \partial_t \mathsf{B}_\mathsf{r} dS = 0\,, \tag{23}$$

the flux through a patch S on the core surface bounded by a contour of zero radial field must be constant. A test of this hypothesis is to compare the changes of the flux through individual flux patches over the time of the model. The patch R2 shows in 17 years (1980 – 1997) a change of its flux of about -20 MWb. This is of the same order of magnitude as the analysis as *Bloxham* [1988], who analysed the same patch for the period 1968.5 to 1980 and found -25 MWb in 11.5 years. Another patch which shows drastic increase of flux is the St. Helena patch R3: within 20 years it changed its flux by 40 MWb, also in agreement with *Bloxham* [1988]. In the northern hemisphere, the North Pole patches show an overall change in flux of about 43 MWb. Further, the merging of the patches R1 and R2 and the splitting of patch R8, both violate frozen flux.

Beside the analysis of the flux through individual flux patches, there is one further possibility to test the consistency of the model, and consequently of the data, with the frozen flux hypothesis. A necessary, but not sufficient, condition for the frozen flux to apply is that the quantity of the unsigned flux integral

$$\int_{CMB} |B_r| dS = \text{const.}\,, \tag{24}$$

integrated over the surface of the core-mantle boundary should not change with time [*Bondi and Gold*, 1950]. Fig. 10 shows the rate of change of the unsigned flux integral (per year) for individual epochs within the modelled period, where we have truncated our model at different harmonic degrees. Closest agreement with frozen flux is obtained for truncation degrees of 11 or 12, which, perhaps by coincidence, is the harmonic degree to which we believe our model to have content aside from the constraints of the end-point models, as discussed above.

Holme and Olsen [2006] computed the unsigned flux integral as here, but using models of the secular variation and main field based on satellite data. Their results for unsigned flux integral concur with those here discussed up to truncation degree 9 or 10. Beyond that the integrals differ due to different modelling approaches. As our resolution improves to include higher degree field components, the unsigned flux integral is more closely conserved suggesting a global agreement with the frozen flux assumption. However, as argued by *Holme and Olsen* [2006], the "blue" nature of the spectrum of the SV at the CMB in models from satellite data alone suggests that no test of frozen-flux can currently be conclusive. Because each successive degree of SV has higher power than the previous one, the calculation of the unsigned flux is inherently unstable. Further, individual small scale features (the reverse flux patches) reveal strong evidence of the violation of the frozen flux assumption, and therefore the presence of diffusion.

overall, the C^{3}FM model can indicate diffusion and a violation of the frozen flux hypothesis even on sub-decadal time scales, but it does not inevitably mean that the frozen flux assumption has to be abandoned, rather that the assumption is inadequate to explain the resolved secular variation entirely. In particular, the observed secular variation in the southern hemisphere is likely to be partly due to diffusion.

7. Conclusion

In this paper, a time–dependent model of the secular variation parameterized up to degree and order 15 has been developed for the period 1980 to 2000. The model is constrained at the end points by satellite models derived from MAGSAT, CHAMP and Ørsted data. It is unique in the least squares sense that it minimizes the model norms for the chosen damping parameters subject to fit the data. The results of the resolution analysis suggest that the constructed model is resolved by the data at least up to degree 12 for the main field and up to degree 10 for its secular variation. Treatment of error covariance combined with a high temporal density of basis splines allows detailed temporal resolution of the field. We therefore feel that our model allows more detailed consideration of the spatio-temporal structure of the core field than either previous models or, the data alone. The model is a sensitive tool to recover the geomagnetic jerks that occurred in this period, and even resolve the changes of flux through the core–mantle boundary. Further, it provides evidence for a violation of the frozen flux hypothesis and therefore facilitates a valuable test of this hypothesis, which is widely applied in the computation of core surface motion.

Acknowledgments. The authors would like to record their gratitude to S. Macmillan, N. Olsen and M. Mandea for providing data and many discussions. All maps of this article were generated with a free software, GMT [*Wessel and Smith*, 1991]. The work was funded by the Deutsche Forschungsgemeinschaft Schwerpunktprojekt 1097.

Notes

1. http://web.dmi.dk/projects/wdcc1/master.html

2. http://www.intermagnet.org/myservlet/imotbl_e.jsp

3. http://www.geomag.bgs.ac.uk/gifs/annual_means.shtml

4. http://www.geomag.bgs.ac.uk/gifs/surveydata.html

References

Alexandrescu, M., D. Gibert, G. Hulot, J. Le Mouël, and G. Saracco (1995), Detection of geomagnetic jerks using wavelet analysis, *J. geophys. Res.*, *100*, 12,557–12,572.

Alfvén, H. (1942), On the existence of electromagnetic-hydromagnetic waves, *Arkiv. Mat. Astron. Fys.*, *29(B)2*, 1–7.

Backus, G. E. (1988), Bayesian inference in geomagnetism, *Geophys. J.*, *92*, 125–142.

Bloxham, J. (1988), The determination of fluid flow at the core surface from geomagnetic observations, in *Mathematical Geophysics, A Survey of Recent Developments in Seismology and Geodynamics*, edited by N. J. Vlaar, G. Nolet, M. J. R. Wortel, and S. A. P. L. Cloetingh, Reidel, Dordrecht.

Bloxham, J., and A. Jackson (1992), Time-dependent mapping of the magnetic field at the core-mantle boundary, *J. geophys. Res.*, *97*, 19,537–19,563.

Bondi, H., and T. Gold (1950), On the generation of magnetism by fluid motion, *Mon. Not. R. Astron. Soc.*, *110*, 607–611.

Box, G. E. P., and G. M. Jenkins (1976), *Time Series Analysis, Forecasting and Control*, Holden Day, San Francisco.

Cain, J. C., Z. Wang, C. Kluth, and D. R. Schmitz (1989), Derivation of a geomagnetic model to $n = 63$, *Geophys. J.*, *97*, 431–441.

de Boor, C. (1978), *A Practical Guide to Splines*, Springer-Verlag, New York.

Dowson, M. J., J. P. Buckingham, and D. A. Simmons (1987), Notes on geomagnetic repeat measurements at Grytviken, April 1987, *Br. Antarc. S. Bull.*, *71*, 49–52.

Gubbins, D. (1975), Can the Earth's magnetic field be sustained by core oscillations?, *Geophys. Res. Lett.*, *2*, 409–412.

Gubbins, D., and J. Bloxham (1985), Geomagnetic field analysis – III. Magnetic fields on the core-mantle boundary, *Geophys. J.*, *80*, 695–713.

Holme, R., and J. Bloxham (1996), The treatment of attitude errors in satellite geomagnetic data, *Phys. Earth Planet. Int*, *98*, 221–233.

Holme, R., and N. Olsen (2006), Core–surface flow modelling from high resolution secular variation, *Geophys. J. Int., in press.*

Jackson, A., A. R. T. Jonkers, and M. R. Walker (2000), Four centuries of geomagnetic secular variation from historical records, *Phil. Trans. R. Soc. Lond. A*, *358*, 957–990.

Kotzé, P. B. (2003), The time–varying geomagnetic field of Southern Africa, *Earth Planets Space*, *55*, 111–116.

Lowes, F. J. (1966), Mean-square values on sphere of spherical harmonic vector fields, *J. geophys. Res.*, *71*, 2179.

Macmillan, S. (1996), A geomagnetic jerk for the early 1990's, *Earth Planet. Sci. Lett.*, *137*, 189–192.

Macmillan S., S. Maus, T. Bondar, A. Chambodut, V. Golovkov, R. Holme, B. Langlais, V. Lesur, F. Lowes, H. Lühr, W. Mai, M. Mandea, N. Olsen, M. Rother, T. Sabaka, A. Thomson, and I. Wardinski (2003), The 9th generation international geomagnetic reference field, *Geophys. J. Int.*, *155*, 1051–1056.

Mandea, M., E. Bellanger, and J.-L. Le–Mouël (2000), A geomagnetic jerk for the end of the 20th century?, *Earth Planet. Sci. Lett.*, *183*, 369–373.

Neubert, T., M. Mandea, G. Hulot, R. von Frese, F. Primdahl, J. L. Jørgensen, E. Friis-Christensen, P. Stauning, N. Olsen, and T. Risbo (2001), Ørsted satellite captures high-precision geomagnetic field data, *Eos Trans. AGU*, *82*, 81–88.

Olsen, N. (2002), A model of the geomagnetic field and its secular variation for epoch 2000 estimated from Ørsted data, *Geophys. J. Int.*, *149*, 455–463.

Priestley, M. B. (1981), *Spectral Analysis and Time Series*, 1. ed., Academic Press and Harcourt Brace Jovanovich.

Reigber, C., H. Lühr, and P. Schwintzer (2002), CHAMP mission status, *Adv. Space Res.*, *30*, 129–134.

Roberts, P. H., and S. Scott (1965), On the analysis of secular variation, 1, A hydromagnetic constraint: Theory, *J. Geomag. Geoelectr.*, *17*, 137–151.

Sabaka, T. J., N. Olsen, and R. A. Langel (2002), A comprehensive model of the quiet-time, near-Earth magnetic field: phase 3, *Geophys. J. Int.*, *151*, 32–68, doi:

10.1046/j.1365-246X.2002.01774.x.

Sabaka, T. J., N. Olsen, and M. E. Purucker (2004), Extending comprehensive models of the Earth's magnetic field with Ørsted and CHAMP data, *Geophys. J. Int.*, *159*, 521–547, doi:10.1111/j.1365-246X.2004.02421.x.

Wessel, P., and W. H. F. Smith (1991), Free software helps map and display data, *Eos Trans. AGU*, *72*, 445–446.

Model parameter	
Number of data	71579
Number of data rejected	5584
RMS misfit at CMB [nT/yr]	700
Number of Splines	13
Damping	
spatial λ_s	8×10^{-18}
temporal λ_t	3×10^{-4}
departure λ_Γ 1980.0	6×10^2
departure λ_Γ 2000.0	6×10^2
Model norms	
spatial N_S [nT^{-2}]	3.58×10^8
temporal N_T [$nT^{-2}yr^{-4}$]	4.02×10^6
departure N_A [$(nT/yr)^2$]	1.93×10^9

Table 1. Model parameters and statistics of the time–dependent modelling

Figure 1. Comparison of the power spectra of the main field at the CMB of GUFM and C^{3}FM.

Figure 2. Comparison of the power spectra of the secular variation at the CMB of GUFM and C^3FM.

Figure 3. Analysis of model residuals for Niemegk. (a) Residuals between model and the secular variation estimates. Residuals for X (long-dashed), Y (short-dashed) and Z (solid) (b) Autocorrelation function of the residuals using the same line styles as for the residuals. (c) Cross-correlation functions of the residuals between $\dot{\mathrm{X}}$ and $\dot{\mathrm{Y}}$ (dashed), between $\dot{\mathrm{X}}$ and $\dot{\mathrm{Z}}$ (solid) and between $\dot{\mathrm{Y}}$ and $\dot{\mathrm{Z}}$ (dot-dashed). The horizontal lines represent the 95 % significance level for not being white noise.

Figure 4. Sub-figures (a) – (c) show a comparison between data (thin line), the interim model (dot-dashed line) and the final model (solid line) for Niemegk, the figures (d) – (f) on the right side show this comparison for the three individual covariance directions using the same line style.

Figure 5. Resolution estimates of the Gauss coefficients at different times. Gauss coefficients are ordered with increasing l and m, i. e. $g_1^0, g_1^1, h_1^1, \ldots, h_{15}^{15}$. The resolution of the coefficients with an order greater than 12 are zero; this indicates a complete control of these coefficients by the a priori beliefs. Resolution at 1980 and 2000 is dominated by the end-model constraints rather than the data

Name	Code	Lat.	Long.	Alt. [m]	N_x	N_y	N_z
Resolute Bay	RES	74.690	265.105	30	232	234	234
Niemegk	NGK	52.072	12.675	78	234	234	234
Newport	NEW	48.263	242.880	780	234	234	236
MBour	MBO	14.392	343.042	5	235	235	235
Patamai	PPT	-17.568	210.425	90	227	227	227
Hermanus	HER	-34.425	19.225	26	235	235	235
Eyrwell	EYR	-43.417	172.350	390	224	224	224
Scott Base	SBA	-77.850	166.783	15	223	223	223

Table 2. List of the observatories shown in fig. 6, their location (Latitude, East longitude, Altitude) and numbers of retained monthly means for X, Y and Z during the model period.

Figure 6. Comparison between the data (thin line), the interim model (dot-dashed line) and the final model (thick solid line) at selected permanent observatories. From left to right: $dX/dt, dY/dt, dZ/dt$. From top to bottom: Resolute Bay (Canada), Niemegk (Germany), Newport (USA), MBour (Senegal), Pamatai (French Polynesia).

Figure 6. *Continued from previous page* From top to bottom: Hermanus (South Africa), Eyrwell (New Zealand), Scott Base (Antarctica).

Figure 7. Radial component of the geomagnetic field at the CMB from C^3FM for four epochs. 1980, 1985, 1995 and 2000 from top to bottom (All maps shown in this article are in Mollweide equal–area projection).

Figure 8. Radial component of the geomagnetic field, with key features of the field: reverse flux patches are labeled R1 – R8, normal flux patches as N1 – N10.

Figure 9. Comparison of the radial field at the CMB from the C^3FM (top) and GUFM (bottom) for 1990.

Patch	Year	Long.	Lat.	Flux [MWb]	$\langle B_r \rangle$ [nT]	Area [km^2]
R1	1980	53.042	-78.746	45.8	81650.4	1.8×10^{12}
	2000	–	–	–	–	–
R2	1980	345.940	-46.246	603.3	137814.3	14.6×10^{12}
	2000	–	–	–	–	–
R3	1980	344.833	-19.600	77.5	80684.6	3.2×10^{12}
	2000	341.263	-20.136	119.4	110936.9	3.5×10^{12}
R4	1980	241.259	-18.106	20.7	34081.6	2.0×10^{12}
	2000	240.895	-17.555	19.3	35121.4	1.8×10^{12}
R5	1980	165.165	19.727	46.1	60198.2	2.5×10^{12}
	2000	165.076	20.109	40.2	56636.3	2.3×10^{12}
R6	1980	316.803	25.524	10.9	34746.5	1.0×10^{12}
	2000	313.089	25.090	6.2	24390.6	0.8×10^{12}
R7	1980	188.700	44.285	16.1	28328.0	1.8×10^{12}
	2000	190.656	45.029	30.8	42026.2	2.4×10^{12}
R8	1980	317.882	80.531	34.9	46351.6	2.5×10^{12}
R8A	2000	31.141	71.053	13.0	32394.2	1.3×10^{12}
R8B	2000	258.085	83.815	65.6	71798.9	3.0×10^{12}

Table 3. Listing of individual reverse flux patches for the epochs 1980 and 2000. The patch R1 merged with R2 in 1986, and R2 itself joined with the flux pattern of the northern hemisphere in 1997.

Figure 10. The difference of the unsigned flux integral between 1980 and 2000 as a function of model truncation. The thick solid represent the averaged rate of change of the unsigned flux integral. The various dashed lines depict the rate of changes of the unsigned flux integral for different epochs.